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Nonlinear Buckling of Thin-Walled Components

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NONLINEAR BUCKLING OF THIN-WALLED COMPONENTS

by

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INTRODUCTION

It is a well-known fact that in the analysis and design of elastic plates, some extra strength beyond the initial buckling load can be utilized safely. The practical significance of this has attracted many studies of the postbuckling behavior of the plates.

In order to deal with elastic postbuckling problem, the geometrically nonlinear effect must be considered. The large deflection introduces not only bending but also membrane deformation. The in-plane load causes not only membrane deformation but also bending. As a result, the coupling of flexural and membrane terms in the plate equations yield some difficulties in resolving them. One of the means of solving such problems is the recently developed finite element method. A comprehensive review of the development on the finite element nonlinear analysis of structures has been presented in Ref. 7.

In this paper, the finite element formulations for the geometrically nonlinear buckling analysis of thin plates with initial deflections is presented. The formulations are appropriate for both the iterative and incremental approaches. The numerical examples are performed by a piecewise linear incremental approach. For economic and practical purpose, the load increments are varied in order to conform to the change of slope of the curve.

ASSUMPTIONS AND POTENTIAL ENERGY

The assumptions underlying this study are:

1. The plate is thin, linear elastic, and isotropic.
2. The deflection is of the same order of magnitude as plate thickness but is still small when compared to other dimensions.
3. The slope is everywhere small in comparison with unity.
4. The in-plane displacements, u and v, are infinitesimal. In the strain displacement relations only those nonlinear terms which depend on $\partial w/\partial x$ and $\partial w/\partial y$ are to be retained. All other nonlinear terms are to be neglected.
5. Kirchhoff's hypotheses hold, i.e., normal stress in the direction transverse to the plate are negligible, and plane transverse sections remain plain after bending.

Based on these assumptions, the strain energy of a differential plate element is of the following form

$$U = \frac{E}{2(1-\nu^2)} \int_V [\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y + (1-\nu)\frac{\epsilon_{xy}^2}{2}]dV \tag{1}$$

where the strains, including the effects of initial deflection and additional large deflection, are given by

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} - z(\frac{\partial^2 w}{\partial x^2}) + \frac{1}{2}[\frac{\partial(w + w_o)}{\partial x}]^2 - \frac{1}{2}(\frac{\partial w_o}{\partial x})^2 \\
\epsilon_y &= \frac{\partial v}{\partial y} - z(\frac{\partial^2 w}{\partial y^2}) + \frac{1}{2}[\frac{\partial(w + w_o)}{\partial y}]^2 - \frac{1}{2}(\frac{\partial w_o}{\partial y})^2 \\
\epsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z(\frac{\partial^2 w}{\partial x \partial y}) + \frac{\partial(w + w_o)}{\partial x}\frac{\partial(w + w_o)}{\partial y} - \frac{\partial w_o}{\partial x}\frac{\partial w_o}{\partial y}
\end{aligned} \tag{2}$$

Substituting the strain-displacement relations given by Eqs. 2 into Eq. 1 , the strain energy of a deformed plate is found

$$U = U_k + U_0 + U_1 + U_2$$

where

$$U_k = \frac{D}{2}\iint \{(\nabla^2 w)^2 + \frac{12}{h^2}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})^2 - 2(1-\nu)[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - (\frac{\partial^2 w}{\partial x \partial y})^2 + \frac{12}{h^2}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{3}{h^2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2]\}dxdy \tag{4}$$

$$U_o = -\frac{3D}{h^2}\iint \{[(\frac{\partial w_o}{\partial x})^2 + \nu(\frac{\partial w_o}{\partial y})^2][\frac{\partial(w + w_o)}{\partial x}]^2 + [(\frac{\partial w_o}{\partial y})^2 + \nu(\frac{\partial w_o}{\partial x})^2][\frac{\partial(w + w_o)}{\partial y}]^2 + 2(1-\nu)\frac{\partial w_o}{\partial x}\frac{\partial w_o}{\partial y}\frac{\partial(w + w_o)}{\partial x}\frac{\partial(w + w_o)}{\partial y}\}dxdy \tag{5}$$

$$U_1 = \frac{6D}{h^2}\iint \{(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y})[\frac{\partial(w + w_o)}{\partial x}]^2 + (\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x})[\frac{\partial(w + w_o)}{\partial y}]^2 + (1-\nu)(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\frac{\partial(w + w_o)}{\partial x}\frac{\partial(w + w_o)}{\partial y}\}dxdy \tag{6}$$

$$U_2 = \frac{3D}{2h^2}\iint \{[\frac{\partial(w + w_o)}{\partial x}]^2 + [\frac{\partial(w + w_o)}{\partial y}]^2\}^2 dxdy \tag{7}$$

In the above energy expression, the first and second order terms of displacement derivatives are disregarded because they do not contribute to the stiffness formulations.

The external work due to the applied loads, p_u, p_v, and p_w acting on the plate in the x, y, and z directions, respectively, is given by

$$W = \int_A (p_u\, u + p_v\, v + p_w\, w)dA \tag{8}$$

The total potential energy of a differential plate element is defined as

$$\Pi = U - W \tag{9}$$

The condition of equilibrium requires that the first variation of the total potential energy vanishes. Based on this requirement, the finite element stiffness formulation can be obtained.

ELEMENT FORMULATIONS

The development of various kinds of finite element for the analysis of plates bending with small deflections is essentially complete. The element shapes may be rectangular, triangular, and quadrilateral, etc.. The nodal degrees of freedom may include the displacements, slopes, curvatures, and "twist derivatives," etc.. The displacement functions may be assumed on the basis of a variety of mathematical functions in order to satisfy the boundary compatibility conditions and to guarantee the fast convergence of solutions. The formulations developed herein are appropriate for any of these finite elements.

Once the finite element model is chosen, the total potential energy expression Π for this element can be obtained by substituting the displacement functions into Eq. 9 or Eqs. 3 and 8. Based on the minimum potential energy theorem, the stiffness equations are obtained by differentiating the total potential energy with respect to each nodal degree of freedom in turn. Adding the contributions of the individual elements, the stiffness equations for the total structure is obtained. These equations are symbolized as of the following matrix form

$$\{P\} = [[K] + [N_o] + [N_1] + [N_2]]\{\Delta + \Delta_o\} \tag{10}$$

where $\{P\}$ and $\{\Delta\}$ are the vectors of nodal loads and displacements, respectively; $[K]$ is the linear stiffness matrix; $[N_o]$ is the zero-order incremental stiffness matrix containing initial displacements; $[N_1]$ is the first-order incremental stiffness matrix containing gross displacement vectors $\{\Delta + \Delta_o\}$; $[N_2]$ is the second-order incremental stiffness matrix containing second-order terms of gross displacements $\{\Delta + \Delta_o\}$; and the

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subscript "o" denotes initial quantities. The matrices $\{P\}$, $[K]$, $[N_o]$, $[N_1]$, and $[N_2]$ are derived from Eqs. 8, 4, 5, 6, and 7, respectively.

Eq. 10 is the formulation appropriate for the direct iterative analysis. The developments concerning the iterative procedure is summarized in Ref. 7.

An alternative solution procedure is the incremental method. The incremental formulation can be obtained by performing Taylor expansion to the equilibrium equation with reference to an equilibrium state. Assuming f is a function of gross displacements $\{\Delta + \Delta_o\}$, its Taylor's expansion about a known equilibrium state $(\{\bar{\Delta} + \Delta_o\}, \{\bar{P}\})$ is given by

$$f_i(\{\bar{\Delta} + \Delta_o\} + \{\delta\Delta\}) = f_i(\{\bar{\Delta} + \Delta_o\}) + \left.\frac{\partial f_i(\{\Delta + \Delta_o\})}{\partial \Delta_j}\right|_{\{\bar{\Delta}\}} \delta\Delta_j + \frac{1}{2} \left.\frac{\partial^2 f_i(\{\Delta + \Delta_o\})}{\partial \Delta_j \partial \Delta_k}\right|_{\{\bar{\Delta}\}} \delta\Delta_j \delta\Delta_k + \cdots \tag{11}$$

where δ denotes an incremental operator.

Applying the Taylor's expansion as defined above to the stiffness equation, Eq. 10, the incremental formulation is obtained. If the last term in Eq. 11 is neglected, the incremental formulation is linearized. It takes the form of

$$\{\delta P\} = [[K] + [N_o] + 2[N_1] + 3[N_2]]\{\delta\Delta\} \tag{12}$$

where the factors 2 and 3 are introduced because the matrices $[N_1]$ and $[N_2]$ are dependent on the linear and quadratic functions of the gross displacement vectors, respectively.

Since the pattern of incremental displacements $\{\delta\Delta\}$ can not be assumed, Eq. 12 can not be applied directly. However, the distribution of load increment is usually known so that Eq. 12 can be applied in an inversed form

$$\{\delta\Delta\} = [[K] + [N_o] + 2[N_1] + 3[N_2]]^{-1}\{\delta P\} \tag{13}$$

in which the matrices $[K]$ and $[N_o]$ are constant at every incremental step; and the incremental stiffness matrices $[N_1]$ and $[N_2]$ are determined by the displacement configuration at the beginning of each incremental step.

It is noted that the matrices $[N_1]$ and $[N_2]$ are given in terms of the undeformed geometry of the plate, the application of Eq. 13 requires no coordinate transformation. However, in the case that the rotations or the slopes $\partial w/\partial x$ and $\partial w/\partial y$ are too large to be neglected, coordinate transformation must then be employed.

The accuracy of this piecewise linear incremental approximation can be increased to any desired level by increasing the number of increments. To choose the proper sizes of load increment is, however, a difficult problem. Undersized increments are wasteful and oversized increments are inaccurate. The error of each incremental step is dominated by the last term in Eq. 11. This term is the second derivative of displacement with respect to load in the load-displacement curve. In order to efficiently reduce the error, the sizes of the load increment must be conform to the variation of the second derivatives. The size must be small when the curvature is large and vice versa. A detail discussion on how to choose the load increments is given by Ref. 10.

TREATING BOUNDARY CONDITION

Unlike the small deflection problems, in a largely deflected plate the edge supporting conditions include not only the flexural conditions but also the membrane conditions. The flexural edge conditions can be directly imposed in Eq. 13 by the standard procedure (8). The membrane edge conditions can, however, not be directly applied in Eq. 13 due to the coupling of flexural and membrane stresses. As can be seen in Eqs. 2, the membrane strains are functions of the first-order terms of the membrane displacement derivatives and the second-order terms of the flexural displacement derivatives. Employing the linear stress-strain relationships and the statically equilibrium conditions, the nodal membrane forces for each element can be written in terms of the first-order nodal membrane displacements and the second-order nodal flexural displacements. The contribution of individual elements are added to provided the nodal membrane force and displacement equation for the whole structure,

$$\{P_m\} = [K_m]\{\Delta_m\} + \lfloor \Delta_f + \Delta_{f_o} \rfloor [S]\{\Delta_f + \Delta_{f_o}\} \tag{14}$$

in which $\{P_m\}$ is the vector of the applied nodal loads in the x-y plane; $[K_m]$ is the membrane portion of the linear stiffness matrix; $[S]$ is a constant matrix; $\{\Delta_m\}$ is the vector of nodal membrane displacements; and $\{\Delta_f + \Delta_{f_o}\}$ is the vector of gross nodal flexural displacements.

Having obtained the gross flexural displacements $\{\Delta_f + \Delta_{f_o}\}$ from Eq. 13, Eq. 14 becomes a linear plane stress problem. The treatment of edge conditions for a plane stress problem is well-known (8). Eq. 14 provides the membrane displacements and membrane stresses at each nodal point.

With the knowledge of the distribution of flexural displacement and the membrane displacement obtained from Eqs. 13 and 14, respectively, the incremental stiffness matrices for a certain equilibrium state is determined and Eq. 13 can be applied for the next incremental step.

POSTBUCKLING TECHNIQUE

The step-by-step linear incremental procedure and the treatment of boundary conditions having been described, the technique of using the procedure to represent postbuckling problems will be considered. Several different methods of reaching the postbuckling state are now suggested:

Method (1)

For predicting the postbuckling behavior of ideally flat plate subjected to edge compression, the following steps are suggested:

(i) The Euler critical load or bifurcation load for a plate is found by solving the lowest eigenvalue of the conventional linear incremental stiffness matrix equation,

$$\text{det.}\left|[K] + [N_1]\right| = 0. \tag{15}$$

the plate is then compressed in the plane by a load slightly above (say 0.2%) the Euler critical load.

(ii) Eq. 13 is not directly applicable because of the intricate membrane-flexure coupling effect. However, Eq. 13 is solvable if the lateral loads rather than middle-surface loads are applied. In order to transforms the middle-surface loads into lateral loads, a slightly deflected shape must be initiated to the flat plate. This can be done either by assuming a distribution of very small initial curvature to the plate or by applying a distribution of very small disturbing lateral load to the plate. It is desirable that the disturbing initial curvature or disturbing lateral load has a similar distribution to the expected postbuckled shape; thus for a single wave the curvature or load distribution is a half since wave, but for a double wave postbuckled shape the load or curvature is distributed as a full sine wave.

(iii) The first-step increments of edge compressive load are applied. This procedure is done by transforming the compressive load increments $\{\Delta P_x\}$, $\{\Delta P_y\}$, $\{\Delta P_{xy}\}$ into lateral load increment $\{\Delta P_z\}$ in the form of

$$\{\Delta P_z\} = \{\Delta P_x\} \frac{\partial^2 w_o}{\partial x^2} + \{\Delta P_y\} \frac{\partial^2 w_o}{\partial y^2} + 2\{\Delta P_{xy}\} \frac{\partial^2 w_o}{\partial x \partial y} \tag{16}$$

and by solving the incremental stiffness formulation of Eq. 13 for flexure displacements.

(iv) Having obtained the flexural displacements, the membrane boundary conditions are imposed and the membrane displacements and stress are found by solving the "plane stress" equations, Eq. 14. Based on the membrane and flexure displacements, the incremental stiffness matrices are determined and the second incremental step is carried out. When applying the second-step increments of in-plane compressive load, they are transformed into the lateral load increment by the form

$$\{\Delta P_z\} = \{\Delta P_x\} \frac{\partial^2 (w + w_o)}{\partial x^2} + \{\Delta P_y\} \frac{\partial^2 (w + w_o)}{\partial y^2} + 2\{\Delta P_{xy}\} \frac{\partial^2 (w + w_o)}{\partial x \partial y} \tag{17}$$

(v) The load increment $\{\Delta P_z\}$ is applied to Eq. 13 for finding flexure displacements. Procedure (iv) is then repeated.

Method (2)

For predicting the postbuckling behavior of plate with small initial curvature subjected to edge compression, the prebuckling behavior is not pronounced and therefore the postbuckling response can be predicted by starting from the Euler critical load rather than the zero load. In this case, Method (1) is followed except that in step (ii) a real distribution of initial curvature of the plate is used rather than an assumed one.

Method (3)

For the case of plate with large initial deflection, the prebuckling load-displacement response is very prominent and is commonly desired. Method (1) still applies provided that steps (i) and (ii) are deleted and the load increments in step (iii) are started from zero loading level.

SNAP-THROUGH PREDICTION

The present piecewise linear incremental procedure is not only appropriate for the prediction of nonlinear load-displacement behavior, but also capable of predicting the snap-through response. In the prediction of the descendent portion of the snap-through curve, the load decrements rather than increments are applied.

In the snap-through predictions, difficulty arises at the maximum and minimum points where the stiffness of the plate vanishes. The singular nature of the stiffness matrices prevents the use of Eq. 13. In order to skip these singular points, two techniques are suggested:

(i) When the maximum or minimum point is approached, a step of displacement-increment is applied to skip the maximum or minimum point. In this step, the displacement pattern is assumed to be the same as that obtained in the previous step.

(ii) When the maximum or minimum point is approached but the slope (or stiffness) of the curve still has a finite value, a step of not-to-small load-increment is applied to skip the maximum or minimum point.

Although both of the above techniques are approximate in nature, the accuracy can be gained by reducing the sizes of those increments which are approaching the maximum or minimum point.

APPLICATIONS

The application of the present formulations and procedures are evaluated and illustrated by a series of examples. If possible, the results are also compared with existing solutions available in the literature.

The finite element model chosen for the following illustrative examples is a rectangular plate element. There are six generalized forces acting at each of the four nodal points. Five of these are the conventional direct forces (3 in number) and bending moments (2 in number). Corresponding to the three direct forces and the two bending moments, are the three direct displacements and the two bending rotations. The sixth generalized force, whose counterpart generalized displacement corresponds to the twist derivative $(\partial^2 w/\partial x \partial y)$, is introduced to assure inclusion of the strain due to simple twist. The membrane displacement functions u and v are chosen as bilinear or first-order Lagrangian interpolation functions which provide linear representation in the x and y directions. The lateral displacement function for w is obtained from the fourth-order Hermitian polynomial approach (2).

Substituting the displacement functions u, v, and w into Eqs. 3 to 8, integrating, and then differentiating with respect to each nodal degree of freedom in turn, a standard force-displacement matrix equation is obtained. Performing first-order Taylor expansion, the equation is converted to the matrix incremental equation.

In order to reduce the computer storage and computing expenses and to increase the questionable accuracy due to inversing the large order stiffness matrix, a reduction technique is used in the following numerical examples. The technique removes about half of the degrees of freedom by expressing them in terms of those degrees of freedom to be retained. The inter-degrees of freedom relationship is derived by the use of linear stiffness matrix [K]. This relationship which reduces the size of the linear stiffness matrix is then applied to reduce the sizes of the incremental stiffness matrices $[N_o]$, $[N_1]$, and $[N_2]$. The concept of such reduction is initiated by that introduced for reducing the mass matrix in the vibration problem (4).

The first example considered is a rectangular flat plate with all edge simply supported with no in-plane movement. The plate is subjected to uniform lateral pressure p. Because of the symmetrical nature of the problem, only one quadrant of the plate is under analysis and 16 finite elements are used. The dimensionless pressure vs. center deflection curves are obtained for the plate with aspect ratio equal to 1 and 1.5, respectively. The results are shown in Fig. 1. The small circles provide an indication of the load increments chosen. The alternative solutions for the same problem are available in Refs. 1 and 6. They are also shown in Fig. 1 for comparison. Close agreement is indicated.

The second example is a square flat plate with all edge simply supported. The edges are also allowed to move in the plane of the plate but are maintained straight by a distribution of normal stress whose resultant is zero. The plate is subjected to uniform lateral pressure. 16 elements are used to idealize one quadrant of the plate. The results for dimenionless pressure vs. center deflection is shown in Fig. 2. The sizes of the load increments are indicated by the small circles. An alternative solution for this problem is available in Ref. 6 by Levy. His solution is also shown in Fig. 2 for comparison. Fair agreement is found.

The third example is an infinitely long rectangular plate with edges simply supported with no in-plane movement. The plate is initially curved with the deflection function

$$\bar{w}_o = W_o \sin \frac{\pi y}{b} \tag{18}$$

The plate is subjected to uniform lateral pressure acting in the same direction as the initial deflection. Because of the symmetrical nature of the problem, only half of a strip is under analysis and 6 elements are used. The results of dimensionless pressure vs. maximum net deflection for the plate with various initial curvatures are shown in Fig. 3. In all of the curves obtained, the load increments are controlled by a geometric series with first increment equal to 6 and each successive increment is greater than the previous one by 1.15 times. A very approximate solution for the problem may be achieved by the use of a single sinusoidal function to represent the deflection shape (9). Such approximate solutions are also shown in Fig. 3 for comparison. Small but consistent discrepancies

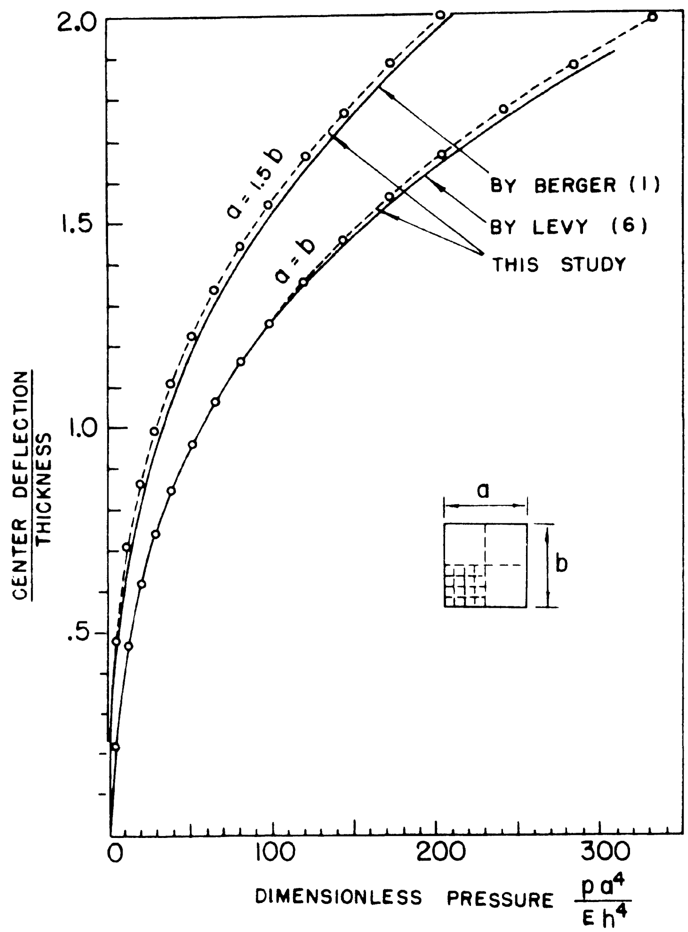

Fig. 1. The load-deflection of rectangular plates with all edges simply supported with no in-plane displacement.

are found in all cases. It is noted that for the case that the plate is flat, an exact solution by Bubnov is also available in Ref. 9. Total agreement between his solution and the present solution is found.

The fourth example is a square flat plate with simply supported edges and subjected to in-plane compression applied at two opposite edges of the plate. Two different membrane edge conditions are considered:

a. The two loaded edges are maintained straight with zero shearing stress. The unloaded edges are maintained straight by a distribution of normal membrane stress, the resultant of which is zero. The edges may move bodily in the plane of the plate and the shearing stress is zero.

b. The two loaded edges are maintained straight with zero shearing stress. the unloaded edges are free to wave in the plane of the plate and the normal membrane stress and shearing stress are zero.

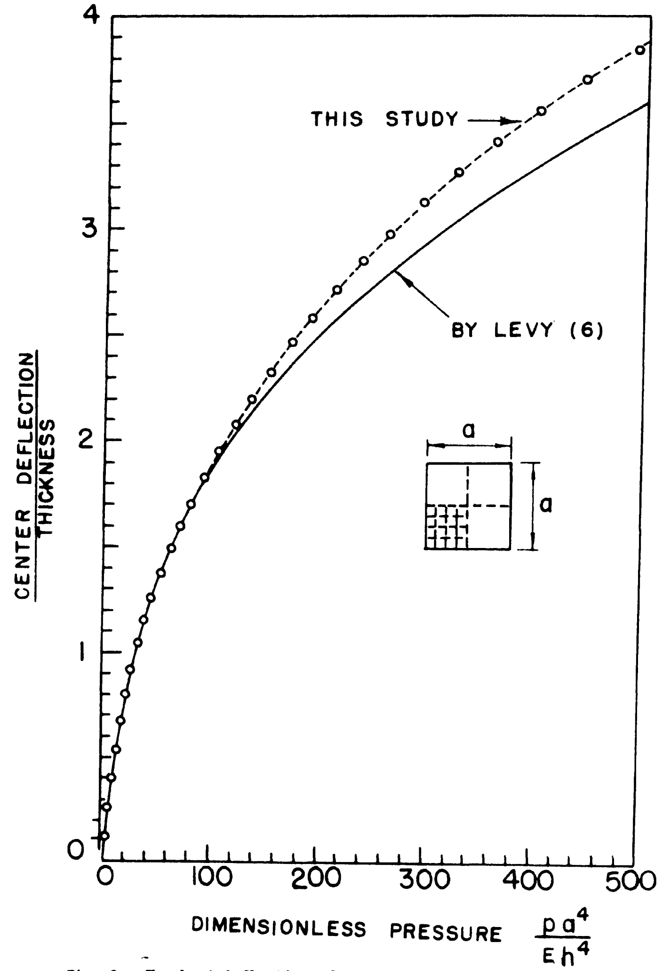

Fig. 2. The load-deflection of a square plate with all edges simply supported and zero edge compression.

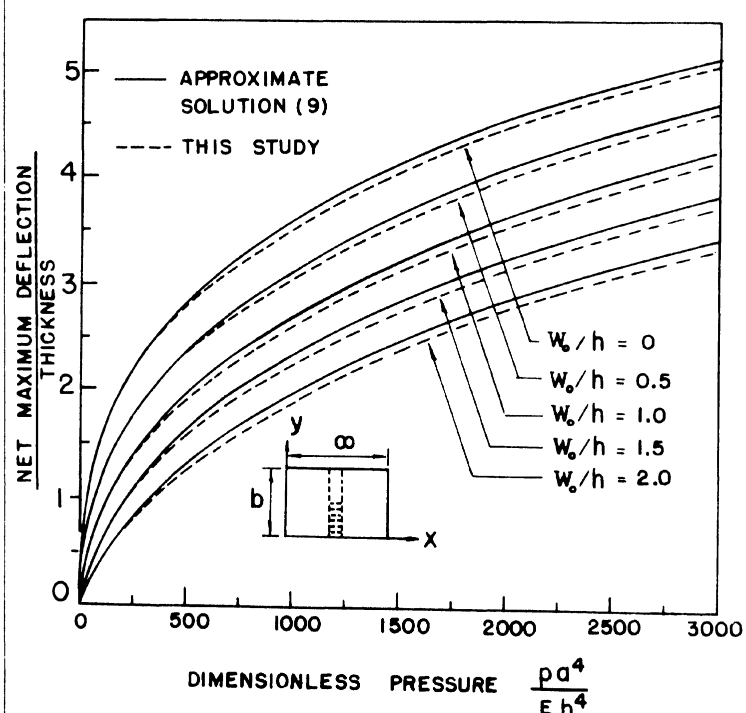

Fig. 3. The load-deflection of a simply-supported long rectangular plate with pressure acting in the same direction of initial deflection.

Because of the symmetrical nature of the problem, only one quadrant of the plate is under analysis and 16 elements are used. The postbuckling load vs. center deflection behavior for both edge conditions (a) and (b) described above are obtained and shown in Fig. 4. The small circles indicate the step sizes used. The alternative series solutions for the same problem are obtained by Levy (6) and Coan (3) for the edge conditions (a) and (b), respectively, Their results are also shown in Fig. 4 for comparison. The agreement is fair.

The fifth example is a square plate with edges simply supported and with membrane edge condition (a). The plate is initially deflected and subjected to in-plane compression applied at two opposite edges. The initial deflection is assumed to be of the following function,

$$\bar{w}_o = W_o \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{19}$$

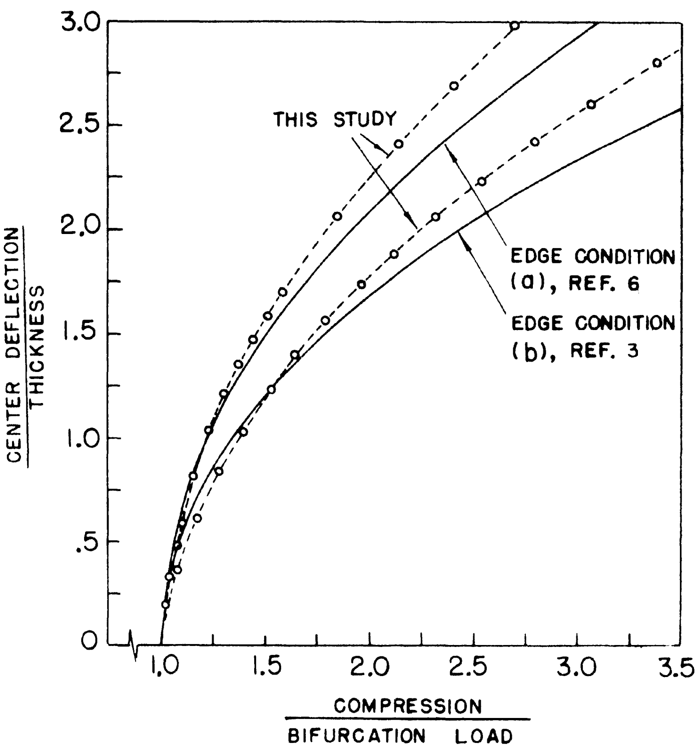

Fig. 4. The postbuckling load-deflection of a square plate with two different edge conditions.

One quadrant of the plate is under analysis and 16 elements are used. The results of dimensionless load vs. net center deflection for the plate with various initial deflections are obtained and shown in Fig. 5. The small circles indicate the step sizes chosen. An alternative series solution for this problem is available in Ref. 5. Those results are also shown in Fig. 5 for comparison. The agreement is close.

The last example is an infinitely long rectangular plate with edges simply supported with no in-plane movement. The plate is initially curved with the deflection shape as given by Eq. 18. The maximum initial deflection is equal to twice the plate thickness, $W_o = 2h$. The plate is subjected to uniform lateral pressure acting in the direction opposite to the initial deflection. This example is a typical snap-through buckling problem. The solution is achieved by using six elements to idealize half of a strip. The results for dimensionless pressure vs. net maximum deflection are plotted in Fig. 6. The small circles indicate the load increments and decrements used. The maximum and minimum points are skipped by applying the small displacement increments. The deflection mode at

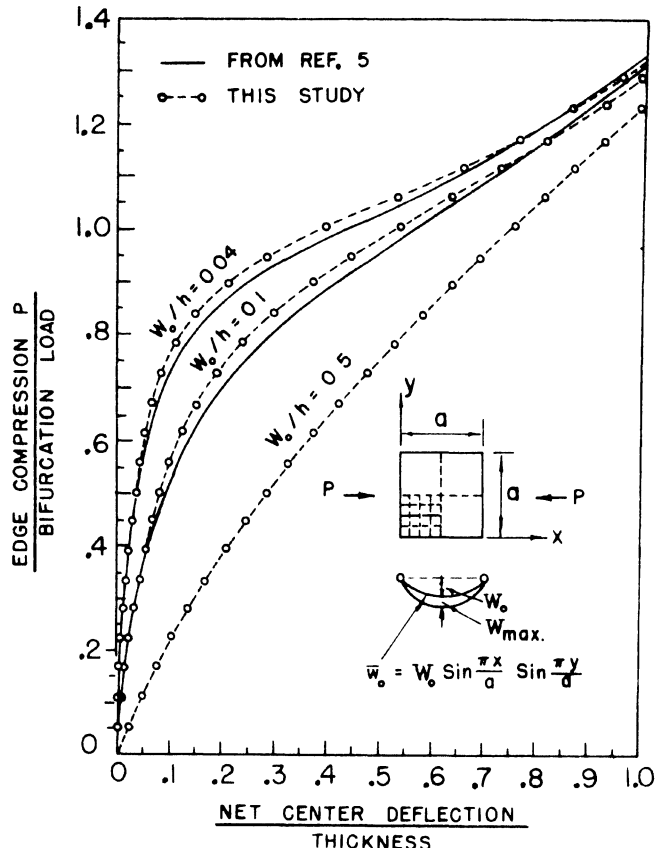

Fig. 5. The load-deflection of a square plate with different initial deflections and with edge conditions (a).

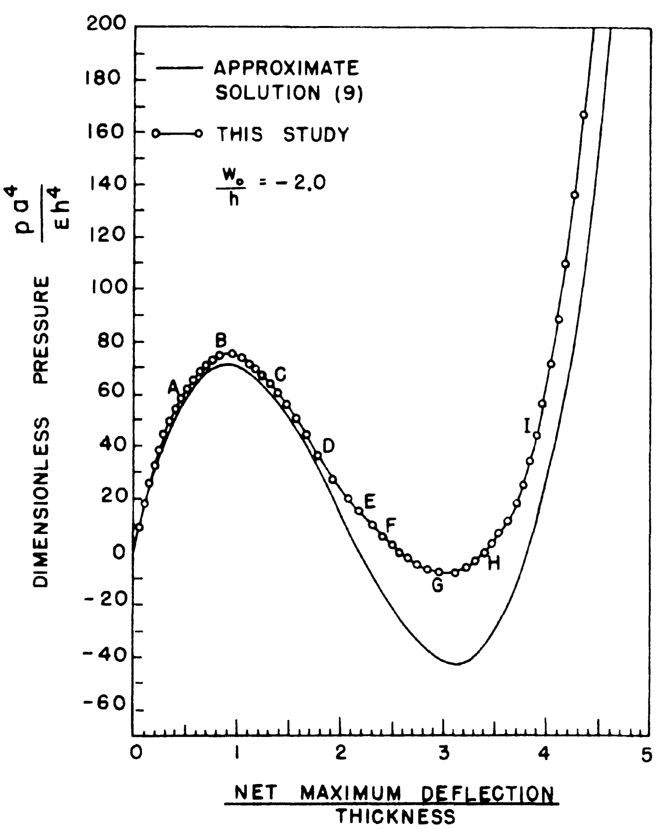

Fig. 6. The snap-buckling load-deflection of a long rectangular plate with pressure acting in the direction opposite to the initial deflection.

different loading stages A, B, ... I are shown in Fig. 7. This problem can also be very approximately solved by using a single sinusoidal function to represent the deflection shape (9). This solution is also shown in

Fig. 6. It is seen that pronounced discrepancy exists in the range of minimum load. This is not surprising simply because the single sinusoidal function can not represent the deflection shape there. In fact, the deflection mode is of multiple wave form in that range as can be seen in Fig. 7.

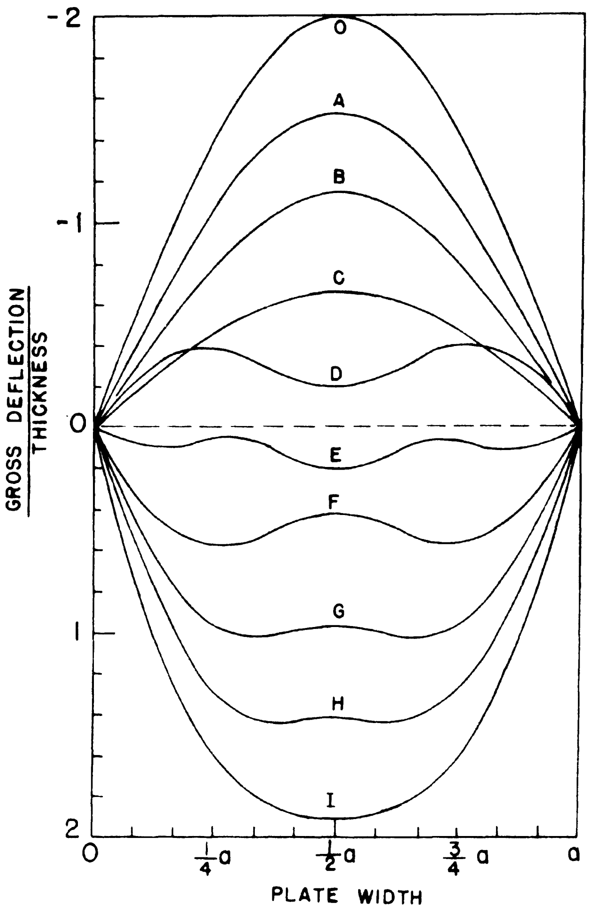

Fig. 7. The deflection modes of the plate in Fig. 6 at different loading stages.

CONCLUSIONS

A finite element formulation for the analysis of geometrically nonlinear buckling response for plates with initial deflections has been presented. The procedural aspects regarding the piecewise linear incremental procedure has been described. A variety of typical plate problems has been examined and the results agree favorably with the known solutions available in the literatures.

From the formulations, procedures, and evaluative analysis shown above, it may be concluded that this study have provided a reasonable background for the extension of the finite element method for treating the large deflection and stability problems of thin-walled structures.

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APPENDIX II. - NOTATION

a,b	= length and width of the rectangular plate;
E	= modulus of elasticity;
h	= thickness of the plate;
[K]	= total system linear stiffness matrix;
$[N_0]$, $[N_1]$, $[N_2]$	= total system incremental stiffness matrices of zero-, first-, and second-order, respectively;
o	= subscript denoting initial quantity;
p	= uniform lateral pressure;
{P}	= nodal load vector of the system;
u, v, w	= components of the displacements;
w_o	= initial deflection of the element;
W_o	= maximum initial deflection of the plate;
$\bar{w}_o$	= initial deflection of the plate;
x, y, z	= global coordinates with the plate in the x-y plane;
{Δ}	= nodal displacement vector of the plate;
δ	= incremental operator;
ν	= Poisson's ratio, for all examples herein, ν = 0.316;
{ }, ⌊ ⌋	= column and row matrices, respectively; and
[ ]	= rectangular matrix.